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Third order difference equations with two rational integrals

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Abstract
A systematic investigation to derive three-dimensional analogs of two-dimensional Quispel, Roberts and Thompson (QRT) mappings is presented. The question of integrability of the obtained three-dimensional mappings with two independent integrals is also analyzed. It is also shown that there exist three-dimensional QRT maps with three n-dependent integrals.

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1. Introduction

The study of nonlinear dynamical systems results in difference equations or mappings that model discrete systems. Discrete systems governed by difference equations are more fundamental than the continuous ones described by differential equations [2, 6]. The theory of differential equations (both ordinary and partial) is reasonably well developed in contrast to the theory of difference equations whose study is only in its initial stage. The theory of special functions has offered interesting examples of nonlinear difference equations and motivated its study. The significance of discrete integrable systems lies in the fact that many areas of mathematics and physics like algebraic geometry, complex analysis, theory of special functions, graph theory, Galois theory, spectral theory and difference geometry have an intimate connection with it.

The study of nonlinear difference equations has drawn much attention from different points of view including its integrability by several researchers [3, 5, 7, 11, 14–16, 25, 26]. Considerable progress have been made for second order ordinary difference equations ($O\Delta E$) or mappings toward finding its solution and analyzing its integrability [4, 12, 18, 19]. Systematic efforts to analyze third order $O\Delta E$ particularly from the point of view of integrability have been made by several researchers in recent years [1, 8, 10, 13, 17, 20–24]. But a more general form of third order $O\Delta E$ of the Quispel, Roberts and Thompson (QRT) form similar to the one in the second order case is elusive. The purpose of the article is to

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explain how to construct three-dimensional analogs of the two-dimensional QRT mappings with one or two rational integrals.

It is appropriate to mention here that there exists no unique definition of integrability either for differential or difference equations. One of the notions of integrability is related with the existence of sufficient number of integrals. An integral (also referred to as conserved quantity) is a function that is not identically constant but is constant on all solutions of the $O\Delta E$. An autonomous Nth order $O\Delta E$ is said to be integrable if (i) it is measure preserving and (ii) admits $N-1$ functionally independent integrals. This working definition provides an effective tool to identify autonomous integrable difference equations.

The plan of the paper is as follows. In section 2, we explain how to construct three-dimensional QRT mappings with one or two rational integrals. In section 3, we give a brief summary of our results. Also the question of integrability of the obtained three-dimensional mappings with two independent integrals is discussed. In appendix A we provide a list of three-dimensional QRT mappings possessing three n-dependent integrals.

2. Construction of a rational integral for third order $O\Delta E$

Consider an autonomous third order $O\Delta E$ having the form

$$w_{n+3} = F(w_n, w_{n+1}, w_{n+2}) \qquad \text{or} \qquad w_3 = F(w_0, w_1, w_2), \qquad w_0 = w_n = w(n). \tag{2.1}$$

A non-trivial function $I(w_n, w_{n+1}, w_{n+2})$ is said to be an integral for (2.1) if $I(w_n, w_{n+1}, w_{n+2}) = I(w_{n+1}, w_{n+2}, w_{n+3})$ holds. In [23], the authors have proposed a method to construct a polynomial integral for (2.1) having the form

$$I(w_0, w_1, w_2) = \sum_{j=1}^{3} \left[A_{1j}(w_1)w_2^2 + A_{2j}(w_1)w_2 + A_{3j}(w_1)\right]w_0^{3-j}$$

and identified several third order $O\Delta E$ possessing two independent integrals. In this paper, we wish to explain how to construct a rational integral for (2.1) having the form

$$I(w_0, w_1, w_2) = \frac{P(w_0, w_1, w_2)}{Q(w_0, w_1, w_2)} = \frac{\sum_{j=1}^{3} \left[A_{1j}(w_1)w_2^2 + A_{2j}(w_1)w_2 + A_{3j}(w_1)\right]w_0^{3-j}}{\sum_{j=1}^{3} \left[a_{1j}(w_1)w_2^2 + a_{2j}(w_1)w_2 + a_{3j}(w_1)\right]w_0^{3-j}}, \tag{2.2}$$

where $A_{ij}(w_1)$ and $a_{ij}(w_1)$ are arbitrary functions. We wish to mention that by considering

$$I(w_0, w_1, w_2) = \frac{\sum_{j=1}^{3} \left[A_{1j}(w_1)w_2^2 + A_{2j}(w_1)w_2 + A_{3j}(w_1)\right]w_0^{3-j}}{w_0 w_1 w_2}, \tag{2.3}$$

we have identified several third order $O\Delta E$ with two rational integrals [22] (see also [8, 10, 21]). In this paper, we consider a more general rational integral given in (2.2). The integrability condition $I(w_n, w_{n+1}, w_{n+2}) = I(w_{n+1}, w_{n+2}, w_{n+3})$ leads to the quadratic equation in w_3 as

$$\begin{aligned} &\left[\left(\sum_{j=1}^{3} A_{1j}(w_2)w_1^{3-j}\right)Q(w_0, w_1, w_2) - \left(\sum_{j=1}^{3} a_{1j}(w_2)w_1^{3-j}\right)P(w_0, w_1, w_2)\right]w_3^2 \\ &\quad + \left[\left(\sum_{j=1}^{3} A_{2j}(w_2)w_1^{3-j}\right)Q(w_0, w_1, w_2) - \left(\sum_{j=1}^{3} a_{2j}(w_2)w_1^{3-j}\right)P(w_0, w_1, w_2)\right]w_3 \\ &\quad + \left[\left(\sum_{j=1}^{3} A_{3j}(w_2)w_1^{3-j}\right)Q(w_0, w_1, w_2) - \left(\sum_{j=1}^{3} a_{3j}(w_2)w_1^{3-j}\right)P(w_0, w_1, w_2)\right] = 0. \end{aligned} \tag{2.4}$$

Note that the above equation can be solved for w_3 in different ways. For example, if $A_{1i}(w_1) = a_{1i}(w_1) = 0, i = 1, 2, 3,$ and $A_{j1}(w_1) = a_{j1}(w_1), j = 2, 3,$ then we obtain the following $O\Delta E$:

$$w_3 = \frac{F_1(w_1, w_2) - F_2(w_1, w_2)w_0}{F_3(w_1, w_2) - F_4(w_1, w_2)w_0} \qquad \text{or} \qquad w_3 = \frac{F_1 - w_0F_2}{F_3 - w_0F_4}, \tag{2.5}$$

which can be viewed as a three-dimensional QRT mapping possessing one integral

$$I(w_0, w_1, w_2) = \frac{[A_{22}(w_1)w_2 + A_{32}(w_1)]w_0 + A_{23}(w_1)w_2 + A_{33}(w_1)}{[a_{22}(w_1)w_2 + a_{32}(w_1)]w_0 + a_{23}(w_1)w_2 + a_{33}(w_1)}, \tag{2.6}$$

where $A_{22}(w_1), A_{23}(w_1), A_{32}(w_1), A_{33}(w_1), a_{22}(w_1), a_{23}(w_1), a_{32}(w_1)$ and $a_{33}(w_1)$ are arbitrary functions. Here

$$F_1 = \left(\sum_{j=2}^{3} a_{3j}(w_2)w_1^{3-j}\right)\left(\sum_{j=2}^{3} A_{j3}(w_1)w_2^{3-j}\right) - \left(\sum_{j=2}^{3} A_{3j}(w_2)w_1^{3-j}\right)\left(\sum_{j=2}^{3} a_{j3}(w_1)w_2^{3-j}\right),$$

$$F_2 = \left(\sum_{j=2}^{3} a_{j2}(w_1)w_2^{3-j}\right)\left(\sum_{j=2}^{3} A_{3j}(w_2)w_1^{3-j}\right) - \left(\sum_{j=2}^{3} A_{j2}(w_1)w_2^{3-j}\right)\left(\sum_{j=2}^{3} a_{3j}(w_2)w_1^{3-j}\right),$$

$$F_3 = \left(\sum_{j=2}^{3} a_{j3}(w_1)w_2^{3-j}\right)\left(\sum_{j=2}^{3} A_{2j}(w_2)w_1^{3-j}\right) - \left(\sum_{j=2}^{3} A_{j3}(w_1)w_2^{3-j}\right)\left(\sum_{j=2}^{3} a_{2j}(w_2)w_1^{3-j}\right),$$

$$F_4 = \left(\sum_{j=2}^{3} a_{2j}(w_2)w_1^{3-j}\right)\left(\sum_{j=2}^{3} A_{j2}(w_1)w_2^{3-j}\right) - \left(\sum_{j=2}^{3} A_{2j}(w_2)w_1^{3-j}\right)\left(\sum_{j=2}^{3} a_{j2}(w_1)w_2^{3-j}\right).$$

Equation (2.4) can also be solved at least in two more distinct ways for w_3 through factorization. For clarity, we discuss them separately as cases 1 and 2.

2.1. Case 1

Equation (2.4) can be factored as

$$\left(w_3 - \frac{1}{w_0}\left[\frac{A_{13}(w_1)w_2^2 + A_{23}(w_1)w_2 + A_{33}(w_1)}{A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2)}\right]\right)\left(w_3 - \left[\frac{f_2(w_1, w_2) - f_3(w_1, w_2)w_0}{f_1(w_1, w_2) - f_2(w_1, w_2)w_0}\right]\right.$$
$$\left.\times\left[\frac{A_{31}(w_2)w_1^2 + A_{32}(w_2)w_1 + A_{33}(w_2)}{A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)}\right]\right) = 0 \tag{2.7}$$

provided the following conditions are satisfied:

$$A_{21}(w_2)w_1^2 + A_{22}(w_2)w_1 + A_{23}(w_2) = A_{12}(w_1)w_2^2 + A_{22}(w_1)w_2 + A_{32}(w_1), \tag{2.8a}$$

$$a_{21}(w_2)w_1^2 + a_{22}(w_2)w_1 + a_{23}(w_2) = a_{12}(w_1)w_2^2 + a_{22}(w_1)w_2 + a_{32}(w_1), \tag{2.8b}$$

$$\frac{[A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2)]}{[A_{13}(w_1)w_2^2 + A_{23}(w_1)w_2 + A_{33}(w_1)]} = \frac{[a_{11}(w_2)w_1^2 + a_{12}(w_2)w_1 + a_{13}(w_2)]}{[a_{13}(w_1)w_2^2 + a_{23}(w_1)w_2 + a_{33}(w_1)]}$$
$$= \frac{[A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)]}{[A_{31}(w_2)w_1^2 + A_{32}(w_2)w_1 + A_{33}(w_2)]} = \frac{[a_{11}(w_1)w_2^2 + a_{21}(w_1)w_2 + a_{31}(w_1)]}{[a_{31}(w_2)w_1^2 + a_{32}(w_2)w_1 + a_{33}(w_2)]}, \tag{2.8c}$$

where

$$\left.\begin{aligned} f_1(w_1,w_2) &= A_2(w_1,w_2)a_3(w_1,w_2) - a_2(w_1,w_2)A_3(w_1,w_2),\\ f_2(w_1,w_2) &= A_3(w_1,w_2)a_1(w_1,w_2) - a_3(w_1,w_2)A_1(w_1,w_2),\\ f_3(w_1,w_2) &= A_1(w_1,w_2)a_2(w_1,w_2) - a_1(w_1,w_2)A_2(w_1,w_2),\\ \left.\begin{aligned} A_i(w_1,w_2) &= \textstyle\sum_{j=1}^{3} A_{ji}(w_1)w_2^{3-j},\\ a_i(w_1,w_2) &= \textstyle\sum_{j=1}^{3} a_{ji}(w_1)w_2^{3-j}, \end{aligned}\right\} & i=1,2,3 \end{aligned}\right\}. \tag{2.9}$$

Obviously (2.7) can be rewritten as

$$w_3 = \frac{1}{w_0}\left[\frac{A_{13}(w_1)w_2^2 + A_{23}(w_1)w_2 + A_{33}(w_1)}{A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2)}\right], \tag{2.10}$$

$$w_3 = \left[\frac{f_2(w_1,w_2) - f_3(w_1,w_2)w_0}{f_1(w_1,w_2) - f_2(w_1,w_2)w_0}\right]\left[\frac{A_{31}(w_2)w_1^2 + A_{32}(w_2)w_1 + A_{33}(w_2)}{A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)}\right] \tag{2.11}$$

or

$$w_3 = \left[\frac{G_1(w_1,w_2) - G_2(w_1,w_2)w_0}{G_3(w_1,w_2) - G_4(w_1,w_2)w_0}\right]. \tag{2.12}$$

It is known that equation (2.10) arises as a reduction of sine-Gordon lattice equation. It is appropriate to mention that the $O\Delta E$ having the form (2.10) have been analyzed by several authors from different points of view including its complete integrability [8, 10, 13, 17, 21, 22].

From (2.11) it is clear that there exist two possibilities and we denote them as cases (1.1) and (1.2) for further discussion.

2.2. Case 1.1

Let us assume that

$$\left[\frac{A_{31}(w_2)w_1^2 + A_{32}(w_2)w_1 + A_{33}(w_2)}{A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)}\right] = 1 \tag{2.13}$$

and so from (2.11) we obtain another form of QRT mapping in three dimensions

$$w_3 = \frac{[A_3a_1 - a_3A_1] - [A_1a_2 - a_1A_2]w_0}{[A_2a_3 - a_2A_3] - [A_3a_1 - a_3A_1]w_0}, \tag{2.14}$$

with 12 parameters (α_j, β_j), $j = 1, 2, 3, 4, 5, 6$, admitting one integral (2.2) where

$$A_1 = A_1(w_1,w_2) = YDZ^t, \qquad A_2 = A_2(w_1,w_2) = YEZ^t, \qquad A_3 = A_3(w_1,w_2) = YFZ^t$$

$$a_1 = a_1(w_1,w_2) = Y\tilde{D}Z^t, \qquad a_2 = a_2(w_1,w_2) = Y\tilde{E}Z^t, \qquad a_3 = a_3(w_1,w_2) = Y\tilde{F}Z^t$$

$$D = \begin{pmatrix}\alpha_1 & \alpha_4 & \alpha_1\\ \alpha_2 & \alpha_5 & \alpha_4\\ \alpha_3 & \alpha_2 & \alpha_1\end{pmatrix}, \qquad E = \begin{pmatrix}\alpha_4 & \alpha_5 & \alpha_2\\ \alpha_5 & \alpha_6 & \alpha_5\\ \alpha_2 & \alpha_5 & \alpha_4\end{pmatrix}, \qquad F = \begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \alpha_4 & \alpha_5 & \alpha_2\\ \alpha_1 & \alpha_4 & \alpha_1\end{pmatrix}$$

$$\tilde{D} = \begin{pmatrix}\beta_1 & \beta_4 & \beta_1\\ \beta_2 & \beta_5 & \beta_4\\ \beta_3 & \beta_2 & \beta_1\end{pmatrix}, \qquad \tilde{E} = \begin{pmatrix}\beta_4 & \beta_5 & \beta_2\\ \beta_5 & \beta_6 & \beta_5\\ \beta_2 & \beta_5 & \beta_4\end{pmatrix}, \qquad \tilde{F} = \begin{pmatrix}\beta_1 & \beta_2 & \beta_3\\ \beta_4 & \beta_5 & \beta_2\\ \beta_1 & \beta_4 & \beta_1\end{pmatrix}$$

$$Z = \begin{bmatrix} w_2^2 & w_2 & 1\end{bmatrix}, Y = \begin{bmatrix} w_1^2 & w_1 & 1\end{bmatrix}$$

and $A_i(w_1, w_2)$, $i = 1, 2, 3$, as in (2.9).

It is straightforward to check that (2.14) is a measure preserving one with measure $\left[a_1(w_1,w_2)w_0^2 + a_2(w_1,w_2)w_0 + a_3(w_1,w_2)\right]^{-1}$. Also the associated integral given in (2.2) has the reversibility property.

An another independent integral for the identified $O\Delta E$ s namely (2.5) and (2.14) can be constructed either by considering a different form other than (2.2) or identifying the parametric restrictions under which (2.5) becomes (2.14) or vice versa. In this paper we proceed with the latter to construct the second integral. As a consequence, we obtain the following QRT type mappings with two independent integrals. They are as follows.

(i) $$w_3 = \frac{f_2(w_1, w_2) + f_1(w_1, w_2)w_0}{f_1(w_1, w_2) - f_2(w_1, w_2)w_0} \quad or \quad w_3 = \frac{f_2 + f_1 w_0}{f_1 - f_2 w_0}, \tag{2.15}$$

where

$$f_1 = \alpha_1^2((1 + w_1w_2)^2 - (w_1 - w_2)^2) + 2\alpha_1\alpha_2(w_1 + w_2)(1 + w_1w_2) + \alpha_2^2\left(1 + w_1^2\right)\left(1 + w_2^2\right),$$
$$f_2 = 2\alpha_1(w_2 - w_1)(\alpha_1 w_1 w_2 + \alpha_2 w_1 + \alpha_2 w_2 + \alpha_1)$$

$$I_1(w_0, w_1, w_2) = \frac{[\alpha_2 w_1 w_2 + \alpha_1(w_2 - w_1) - \alpha_2]w_0 - (\alpha_1 w_1 w_2 + \alpha_2 w_1 + \alpha_2 w_2 + \alpha_1)}{\left(\alpha_2^2 - \alpha_1^2\right)w_1(w_0 w_2 - 1) - \left[2\alpha_1\alpha_2 w_1 + \left(\alpha_1^2 + \alpha_2^2\right)\right](w_0 + w_2)}, \tag{2.16}$$

$$I_2(w_0, w_1, w_2) = \frac{\left[A_1(w_1, w_2)w_0^2 + A_2(w_1, w_2)w_0 + A_3(w_1, w_2)\right]}{\left[a_1(w_1, w_2)w_0^2 + a_2(w_1, w_2)w_0 + a_3(w_1, w_2)\right]}, \tag{2.17}$$

$$A_1(w_1, w_2) = \alpha_1^2(w_2 - w_1)(1 + w_1w_2) - \alpha_2^2(w_1 + w_2)(1 + w_1w_2) + 2\alpha_1\alpha_2(w_2 - w_1)w_2,$$
$$A_2(w_1, w_2) = -f_1(w_1, w_2), \qquad a_2(w_1, w_2) = f_1(w_1, w_2),$$
$$A_3(w_1, w_2) = \alpha_1^2(w_1 - w_2)(1 + w_1w_2) + 2\alpha_1\alpha_2(w_1 - w_2)w_1 - \alpha_2^2(w_1 + w_2)(1 + w_1w_2),$$
$$a_1(w_1, w_2) = \alpha_1^2(w_1 - w_2)(1 + w_1w_2) + \alpha_2^2(w_1 + w_2)(1 + w_1w_2) + 2\alpha_1\alpha_2\left(w_1^2w_2^2 + w_1w_2 + w_1^2 + 1\right),$$
$$a_3(w_1, w_2) = \alpha_1^2(w_2 - w_1)(1 + w_1w_2) + \alpha_2^2(w_1 + w_2)(1 + w_1w_2) + 2\alpha_1\alpha_2\left(1 + w_1w_2 + w_1^2w_2^2 + w_2^2\right).$$

(ii) $$w_3 = \frac{f_2 + f_1 w_0}{f_1 - f_2 w_0}, \qquad \alpha \neq 1, \tag{2.18}$$

$$f_1 = \left[(1 + \alpha^2)w_1^2 + (1 - \alpha^2)w_1 - 2\alpha\right]w_2^2 + \left[(1 - \alpha^2)w_1^2 + 2(1 + \alpha)^2 w_1 + 1 - \alpha^2\right]w_2 - 2\alpha w_1^2 + (1 - \alpha^2)w_1 + \alpha^2 + 1$$
$$f_2 = (\alpha + 1)\left[([\alpha + 1]w_1 + 1 - \alpha)w_2^2 - (\alpha + 1)\left(w_1^2 - 1\right)w_2 + (\alpha - 1)w_1^2 - (\alpha + 1)w_1\right]$$

$$I_1(w_0, w_1, w_2) = \frac{B(w_2w_0 - 1) + C(w_0 + w_2)}{D(w_2w_0 - 1) + E(w_0 + w_2)} \tag{2.19}$$

$$I_2(w_0, w_1, w_2) = \frac{\left[A_1(w_1, w_2)w_0^2 + A_2(w_1, w_2)w_0 + A_3(w_1, w_2)\right]}{\left[a_1(w_1, w_2)w_0^2 + a_2(w_1, w_2)w_0 + a_3(w_1, w_2)\right]}, \tag{2.20}$$

where

$$B = (\alpha^4 - 4\alpha^2 - 1)w_1 - (1 + \alpha^2)^2, \qquad C = 4\alpha^3 w_1 - 2\alpha^3 - 2\alpha$$
$$D = (\alpha^2 - 1)w_1 - \alpha^2 - 1, \qquad E = 2\alpha w_1$$
$$A_1(w_1, w_2) = \left[(\alpha^4 - 4\alpha^2 - 1)w_1^2 + 4\alpha^3 w_1 - (\alpha^2 + 1)^2\right]w_2^2 + \left[-2\alpha^2(\alpha^2 + 1)w_1^2 + 2\alpha^2(\alpha^2 - 1)w_1 + 4\alpha^3\right]w_2 + (-1 - 4\alpha^2 + \alpha^4)w_1^2 - 2\alpha^2(1 + \alpha^2)w_1 - 1 - 4\alpha^2 + \alpha^4,$$

$$
\begin{aligned}
A_2(w_1, w_2) = {} & \left[2\alpha^2(\alpha^2-1)w_1 - 2\alpha^2(\alpha^2+1)w_1^2 + 4\alpha^3\right]w_2^2 \\
& + \left[2\alpha^2(\alpha^2-1)w_1^2 - 4\alpha^2(\alpha+1)^2 w_1 + 2\alpha^2(\alpha^2-1)\right]w_2 \\
& + 4\alpha^3 w_1^2 + 2\alpha^2(\alpha^2-1)w_1 - 2\alpha^2(\alpha^2+1), \\
A_3(w_1, w_2) = {} & \left[(-4\alpha^2+\alpha^4-1)w_1^2 - 2\alpha^2(\alpha^2+1)w_1 + \alpha^4 - 4\alpha^2 - 1\right]w_2^2 \\
& + \left[4\alpha^3 w_1^2 + 2\alpha^2(\alpha^2-1)w_1 - 2\alpha^2(\alpha^2+1)\right]w_2 \\
& - (\alpha^2+1)^2 w_1^2 + 4\alpha^3 w_1 + \alpha^4 - 4\alpha^2 - 1, \\
a_1(w_1, w_2) = {} & \left[(\alpha^2+1)^2 w_1^2 + 4\alpha^3 w_1 - \alpha^4 + 4\alpha^2 + 1\right]w_2^2 + \left[-2\alpha^2(\alpha^2+1)w_1^2\right. \\
& \left. + 2\alpha^2(\alpha^2-1)w_1 + 4\alpha^3\right]w_2 + (\alpha^2+1)^2 w_1^2 \\
& - 2\alpha^2(\alpha^2+1)w_1 + (\alpha^2+1)^2, \\
a_3(w_1, w_2) = {} & \left[(\alpha^2+1)^2 w_1^2 - 2\alpha^2(\alpha^2+1)w_1 + (\alpha^2+1)^2\right]w_2^2 \\
& + \left[4\alpha^3 w_1^2 + 2\alpha^2(\alpha^2-1)w_1 - 2\alpha^2(\alpha^2+1)\right]w_2 \\
& + (4\alpha^2 - \alpha^4 + 1)w_1^2 + 4\alpha^3 w_1 + (\alpha^2+1)^2, \\
a_2(w_1, w_2) = {} & A_2(w_1, w_2).
\end{aligned}
$$

(iii)
$$w_3 = \frac{w_2 - w_1 + (1 + w_1 w_2)w_0}{1 + w_1 w_2 - (w_2 - w_1)w_0} \tag{2.21}$$

and

$$I_1(w_0, w_1, w_2) = \frac{w_2 - 1 + (1 + w_2)w_0}{w_2 w_0 - 1}, \tag{2.22}$$

$$I_2(w_0, w_1, w_2) = \frac{\left[A_1(w_1, w_2)w_0^2 + A_2(w_1, w_2)w_0 + A_3(w_1, w_2)\right]}{\left[a_1(w_1, w_2)w_0^2 + a_2(w_1, w_2)w_0 + a_3(w_1, w_2)\right]}, \tag{2.23}$$

where

$$
\begin{aligned}
A_1(w_1, w_2) &= w_1^2 w_2 - w_2^2 + w_1 w_2 + w_1, \\
A_2(w_1, w_2) &= a_2(w_1, w_2) = \left(w_1^2 + w_1\right)w_2^2 + (w_1+1)^2 w_2 + 1 + w_1, \\
A_3(w_1, w_2) &= w_1 w_2^2 + (1 + w_1)w_2 - w_1^2, \\
a_1(w_1, w_2) &= w_1^2 w_2^2 + \left(w_1^2 + w_1\right)w_2 + w_1^2 + w_1 + 1, \\
a_3(w_1, w_2) &= \left(w_1^2 + w_1 + 1\right)w_2^2 + (1 + w_1)w_2 + 1.
\end{aligned}
$$

We would like to mention that (2.21) is also a periodic recurrence equation and hence two integrals both quadratic in w_0, w_1, w_2 can be found [9]. Also (2.21) can be transformed into the following linear $O\Delta E$:

$$\theta(n+3) - \theta(n+2) + \theta(n+1) - \theta(n) = p\pi, \qquad \theta(n) = \arctan(w(n)), \tag{2.24}$$

and so the general solution of the identified difference equation (2.21) reads

$$w(n) = \tan\left(c_1 + c_2\cos n + c_3\sin n + \frac{p\pi n}{2}\right), \qquad p \in Z, \tag{2.25}$$

where c_1, c_2 and c_3 are arbitrary constants.

(iv)
$$w_3 = \frac{f_2 + f_1 w_0}{f_1 - f_2 w_0}, \tag{2.26}$$

$$
\begin{aligned}
f_2 &= (w_2 - w_1)\left(3w_2^2 w_1^2 + 8w_1 w_2 - w_1^2 - w_2^2 + 3\right), \\
f_1 &= (1 + w_1 w_2)\left(w_1^2 w_2^2 + 8w_1 w_2 - 3w_1^2 - 3w_2^2 + 1\right). \\
I_1 &= \frac{(w_2 - w_1)(1 + w_0 w_1) - (w_1 - w_0)(1 + w_1 w_2)}{(1 + w_0 w_1)(1 + w_1 w_2) + (w_2 - w_1)(w_1 - w_0)}.
\end{aligned}
$$

Note that (2.26) can be transformed into a linear $O\Delta E$

$$\theta(n+3) - 3\theta(n+2) + 3\theta(n+1) - \theta(n) = p\pi, \qquad \theta(n) = \arctan(w(n)) \tag{2.27}$$

with the general solution

$$w(n) = \tan\left(\frac{p\pi}{6}n^3 + an^2 + bn + c\right), \qquad p \in Z, \tag{2.28}$$

where a, b and c are arbitrary constants.

2.3. Case 1.2

Let us assume that

$$\left[\frac{A_{31}(w_2)w_1^2 + A_{32}(w_2)w_1 + A_{33}(w_2)}{A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)}\right] \neq 1. \tag{2.29}$$

Here again we obtain a third order $O\Delta E$ (2.11) with one integral given by (2.2) with the following explicit forms of A_{ij}'s:

$$A_{11}(w_1) = (\alpha_1 w_1 + \alpha_2)(\beta_1 w_1 + \beta_2), \tag{2.30a}$$

$$A_{21}(w_1) = A_{12}(w_1) = (\alpha_1\beta_2 w_1 + \alpha_1\beta_1 w_1 + \alpha_1\beta_2 + \alpha_2\beta_2)(1 + w_1), \tag{2.30b}$$

$$A_{31}(w_1) = A_{13}(w_1) = \alpha_1\beta_2(1 + w_1)^2, \tag{2.30c}$$

$$A_{22}(w_1) = (\alpha_2\beta_2 + 2\alpha_1\beta_2 + \alpha_1\beta_1)w_1^2 + \alpha_3 w_1 + 2\alpha_1\beta_2 + \alpha_1\beta_3 + \alpha_2\beta_2, \tag{2.30d}$$

$$A_{32}(w_1) = A_{23}(w_1) = (\alpha_1\beta_2 w_1 + \alpha_2\beta_2 w_1 + \alpha_1\beta_2 + \alpha_1\beta_3)(1 + w_1), \tag{2.30e}$$

$$A_{33}(w_1) = (\alpha_2 w_1 + \alpha_1)(\beta_2 w_1 + \beta_3). \tag{2.30f}$$

$$a_{11}(w_1) = (\gamma_1 w_1 + \gamma_2)(\beta_1 w_1 + \beta_2), \tag{2.31a}$$

$$a_{21}(w_1) = a_{12}(w_1) = (\gamma_1\beta_2 w_1 + \gamma_1\beta_1 w_1 + \gamma_1\beta_2 + \gamma_2\beta_2)(1 + w_1), \tag{2.31b}$$

$$a_{31}(w_1) = a_{13}(w_1) = \gamma_1\beta_2(1 + w_1)^2, \tag{2.31c}$$

$$a_{22}(w_1) = (\gamma_2\beta_2 + 2\gamma_1\beta_2 + \gamma_1\beta_1)w_1^2 + \gamma_3 w_1 + 2\gamma_1\beta_2 + \gamma_1\beta_3 + \gamma_2\beta_2, \tag{2.31d}$$

$$a_{32}(w_1) = a_{23}(w_1) = (\gamma_1\beta_2 w_1 + \gamma_2\beta_2 w_1 + \gamma_1\beta_2 + \gamma_1\beta_3)(1 + w_1), \tag{2.31e}$$

$$a_{33}(w_1) = (\gamma_2 w_1 + \gamma_1)(\beta_2 w_1 + \beta_3). \tag{2.31f}$$

It is not clear to us how to construct the second independent integral for (2.11) in general and is under investigation.

2.4. Case 2

Equation (2.4) can also be factored into

$$\left(w_3 - w_0\left[\frac{A_{31}(w_2)w_1^2 + A_{32}(w_2)w_1 + A_{33}(w_2)}{A_{13}(w_1)w_2^2 + A_{23}(w_1)w_2 + A_{33}(w_1)}\right]\right)\left(w_3 - \left[\frac{f_1(w_1,w_2) - w_0 f_2(w_1,w_2)}{f_2(w_1,w_2) - w_0 f_3(w_1,w_2)}\right]\right.$$
$$\left.\times\left[\frac{A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)}{A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2)}\right]\right) = 0 \tag{2.32}$$

provided in addition to (2.8a)–(2.8b) the following condition is satisfied:

$$\frac{[A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)]}{[A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2)]} = \frac{[A_{31}(w_2)w_1^2 + A_{32}(w_2)w_1 + A_{33}(w_2)]}{[A_{13}(w_1)w_2^2 + A_{23}(w_1)w_2 + A_{33}(w_1)]}$$
$$= \frac{[a_{31}(w_2)w_1^2 + a_{32}(w_2)w_1 + a_{33}(w_2)]}{[a_{13}(w_1)w_2^2 + a_{23}(w_1)w_2 + a_{33}(w_1)]} = \frac{[a_{11}(w_1)w_2^2 + a_{21}(w_1)w_2 + a_{31}(w_1)]}{[a_{11}(w_2)w_1^2 + a_{12}(w_2)w_1 + a_{13}(w_2)]}, \tag{2.33}$$

where $f_i(w_1, w_2), i = 1, 2, 3$, are given in (2.9). As before (2.32) can be rewritten as

$$w_3 = w_0 \left[\frac{A_{31}(w_2)w_1^2 + A_{32}(w_2)w_1 + A_{33}(w_2)}{A_{13}(w_1)w_2^2 + A_{23}(w_1)w_2 + A_{33}(w_1)} \right], \tag{2.34}$$

$$w_3 = \left[\frac{f_1(w_1, w_2) - w_0 f_2(w_1, w_2)}{f_2(w_1, w_2) - w_0 f_3(w_1, w_2)} \right] \left[\frac{A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)}{A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2)} \right] \tag{2.35}$$

or

$$w_3 = \left[\frac{\tilde{G}_1(w_1, w_2) - \tilde{G}_2(w_1, w_2)w_0}{\tilde{G}_3(w_1, w_2) - \tilde{G}_4(w_1, w_2)w_0} \right]. \tag{2.36}$$

Note that (2.34) arises as a reduction of the modified K-dV lattice equation and has been studied extensively by many authors [8, 10, 13, 17, 21, 22]. From (2.35) it is clear that there exist two possibilities and we denote them as cases (2.1) and (2.2) for further discussion.

2.5. *Case 2.1*

Let us assume that

$$\frac{A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)}{A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2)} = 1 \tag{2.37}$$

and so (2.35) gives another QRT type mapping in three dimensions

$$w_3 = \left[\frac{f_1(w_1, w_2) - f_2(w_1, w_2)w_0}{f_2(w_1, w_2) - f_3(w_1, w_2)w_0} \right] \tag{2.38}$$

with 22 parameters admitting one integral (2.2) which is cyclic invariant where

$$A_1(w_1, w_2) = YDZ^t, \qquad A_2(w_1, w_2) = YEZ^t, \qquad A_3(w_1, w_2) = YFZ^t$$
$$a_1(w_1, w_2) = Y\tilde{D}Z^t, \qquad a_2(w_1, w_2) = Y\tilde{E}Z^t, \qquad a_3(w_1, w_2) = Y\tilde{F}Z^t$$

and

$$D = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & \alpha_4 & \alpha_5 \\ \alpha_3 & \alpha_6 & \alpha_7 \end{pmatrix}, \qquad E = \begin{pmatrix} \alpha_2 & \alpha_4 & \alpha_6 \\ \alpha_4 & \alpha_8 & \alpha_9 \\ \alpha_5 & \alpha_9 & \alpha_{10} \end{pmatrix}, \qquad F = \begin{pmatrix} \alpha_3 & \alpha_5 & \alpha_7 \\ \alpha_6 & \alpha_9 & \alpha_{10} \\ \alpha_7 & \alpha_{10} & \alpha_{11} \end{pmatrix}$$

$$\tilde{D} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_4 & \beta_5 \\ \beta_3 & \beta_6 & \beta_7 \end{pmatrix}, \qquad \tilde{E} = \begin{pmatrix} \beta_2 & \beta_4 & \beta_6 \\ \beta_4 & \beta_8 & \beta_9 \\ \beta_5 & \beta_9 & \beta_{10} \end{pmatrix}, \qquad \tilde{F} = \begin{pmatrix} \beta_3 & \beta_5 & \beta_7 \\ \beta_6 & \beta_9 & \beta_{10} \\ \beta_7 & \beta_{10} & \beta_{11} \end{pmatrix}$$

$$Z = \begin{bmatrix} w_2^2 & w_2 & 1 \end{bmatrix}, \qquad Y = \begin{bmatrix} w_1^2 & w_1 & 1 \end{bmatrix}$$

and $A_i(w_1, w_2), i = 1, 2, 3$, as in (2.9). It is easy to check that (2.38) is a measure preserving one with the measure $\left[a_1(w_1, w_2)w_0^2 + a_2(w_1, w_2)w_0 + a_3(w_1, w_2)\right]^{-1}$.

The second integral $I_2(w_0, w_1, w_2)$ can be constructed for (2.38) by finding the conditions on the parameters α_i and β_i at which (2.38) becomes (2.5) or vice versa. As a result we obtain a third order $O\Delta E$

$$w_3 = \frac{w_2 + w_1 - (1 - w_1 w_2)w_0}{1 - w_1 w_2 + (w_1 + w_2)w_0}, \tag{2.39}$$

admitting two independent integrals $I_1(w_0, w_1, w_2)$ and $I_2(w_0, w_1, w_2)$:

$$I_1(w_0, w_1, w_2) = \frac{1 + w_2 w_0}{w_2 - 1 - (1 + w_2)w_0}, \tag{2.40}$$

$$I_2(w_0, w_1, w_2) = \frac{B(w_0, w_1, w_2) + C(w_0, w_1, w_2)}{B(w_0, w_1, w_2) - C(w_0, w_1, w_2)}, \tag{2.41}$$

$$B(w_0, w_1, w_2) = \left(w_1^2 + 1\right)\left(w_2^2 + 1\right)\left(w_0^2 + 1\right),$$
$$C(w_0, w_1, w_2) = (w_2 - w_1)\left(w_0^2 - (w_2 + w_1)w_0 + w_1 w_2\right).$$

Also (2.39) can be transformed into a linear $O\Delta E$

$$\theta(n+3) - \theta(n+2) - \theta(n+1) + \theta(n) = p\pi, \qquad \theta(n) = \arctan(w(n)) \tag{2.42}$$

with general solution

$$w(n) = \tan\left(c_1 + c_2 n + c_3(-1)^n + \frac{p\pi n^2}{4}\right), \qquad p \in Z, \tag{2.43}$$

where c_1, c_2 and c_3 are arbitrary constants.

2.6. *Case 2.2*

Let us assume that

$$\frac{A_{11}(w_1)w_2^2 + A_{21}(w_1)w_2 + A_{31}(w_1)}{A_{11}(w_2)w_1^2 + A_{12}(w_2)w_1 + A_{13}(w_2)} \neq 1. \tag{2.44}$$

Then we obtain a third order $O\Delta E$ (2.36) with an integral given by (2.2) where the explicit forms of A_{ij}s are given in (2.30a)–(2.30f) and a_{ij}'s are as follows:

$$a_{11}(w_1) = (\alpha_1 w_1 + \alpha_2)(\gamma_1 w_1 + \gamma_2), \tag{2.45a}$$

$$a_{21}(w_1) = a_{12}(w_1) = (\alpha_1\gamma_2 w_1 + \alpha_1\gamma_1 w_1 + \alpha_1\gamma_2 + \alpha_2\gamma_2)(1 + w_1), \tag{2.45b}$$

$$a_{31}(w_1) = a_{13}(w_1) = \alpha_1\gamma_2(1 + w_1)^2, \tag{2.45c}$$

$$a_{22}(w_1) = (\alpha_2\gamma_2 + 2\alpha_1\gamma_2 + \alpha_1\gamma_1)w_1^2 + \gamma_4 w_1 + 2\alpha_1\gamma_2 + \alpha_1\gamma_3 + \alpha_2\gamma_2, \tag{2.45d}$$

$$a_{32}(w_1) = a_{23}(w_1) = (\alpha_1\gamma_2 w_1 + \alpha_2\gamma_2 w_1 + \alpha_1\gamma_2 + \alpha_1\gamma_3)(1 + w_1), \tag{2.45e}$$

$$a_{33}(w_1) = (\alpha_2 w_1 + \alpha_1)(\gamma_2 w_1 + \gamma_3). \tag{2.45f}$$

The construction of the second independent integral for (2.36) is under investigation.

3. Conclusion

In this paper, a systematic investigation to derive three-dimensional analogs of two-dimensional QRT mappings is presented. We have identified four distinct third order $O\Delta E$s namely (2.15), (2.18), (2.21) and (2.39) admitting two independent rational integrals. Moreover each of them is a measure preserving one. We have observed that one of the rational integrals in each of the obtained mappings is tri-quadratic. Note that the mapping given in (2.26) admits one rational integral and can be transformed into a linear $O\Delta E$ through a global transformation. In appendix A we provide a list of three distinct QRT mappings in three dimensions possessing three n-dependent integrals. Hence, we conclude that all the identified eight cases of third order $O\Delta E$ with two or three integrals are integrable. The existence of the second integral for the maps given in cases (1.2) and (2.2) is under investigation.

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Appendix A. Third order QRT maps with *n*-dependent integrals

We would like to report that we have obtained the following QRT maps (through the trial and error method) admitting three independent n-dependent integrals:

$$w_3 = \left[\frac{f_1(w_1, w_2) - f_2(w_1, w_2)w_0}{f_2(w_1, w_2) - f_3(w_1, w_2)w_0}\right] \tag{A.1}$$

(i) $f_1(w_1, w_2) = \lambda_2{}^2 P(w_2 - \lambda_1)^k (w_1 - \lambda_1)^l - \lambda_1{}^2 (w_2 - \lambda_2)^k (w_1 - \lambda_2)^l,$
$f_2(w_1, w_2) = \lambda_2 P(w_2 - \lambda_1)^k (w_1 - \lambda_1)^l - \lambda_1 (w_2 - \lambda_2)^k (w_1 - \lambda_2)^l,$
$f_3(w_1, w_2) = P(w_2 - \lambda_1)^k (w_1 - \lambda_1)^l - (w_2 - \lambda_2)^k (w_1 - \lambda_2)^l.$

$$I_1 = \alpha_1^{-n}\left\{\log\left[\left(\frac{w_2-\lambda_1}{w_2-\lambda_2}\right)\left(\frac{w_1-\lambda_2}{w_1-\lambda_1}\right)^{\alpha_2+\alpha_3}\left(\frac{w_0-\lambda_1}{w_0-\lambda_2}\right)^{\alpha_2\alpha_3}\right] + \frac{(1-\alpha_2)(1-\alpha_3)}{k+l-2}\log P\right\},$$
$$I_2 = \alpha_2^{-n}\left\{\log\left[\left(\frac{w_2-\lambda_1}{w_2-\lambda_2}\right)\left(\frac{w_1-\lambda_2}{w_1-\lambda_1}\right)^{\alpha_1+\alpha_3}\left(\frac{w_0-\lambda_1}{w_0-\lambda_2}\right)^{\alpha_1\alpha_3}\right] + \frac{(1-\alpha_1)(1-\alpha_3)}{k+l-2}\log P\right\},$$
$$I_3 = \alpha_3^{-n}\left\{\log\left[\left(\frac{w_2-\lambda_1}{w_2-\lambda_2}\right)\left(\frac{w_1-\lambda_2}{w_1-\lambda_1}\right)^{\alpha_1+\alpha_2}\left(\frac{w_0-\lambda_1}{w_0-\lambda_2}\right)^{\alpha_1\alpha_2}\right] + \frac{(1-\alpha_1)(1-\alpha_2)}{k+l-2}\log P\right\}.$$

(ii) $f_1(w_1, w_2) = \alpha^2 DQ(\alpha - w_2)(\alpha - w_1) - 2\alpha(\alpha - w_2)(\alpha - w_1)$
$+ l\alpha^2(\alpha - w_2) + k\alpha^2(\alpha - w_1),$
$f_2(w_1, w_2) = \alpha K(\alpha - w_1) + \alpha l(\alpha - w_2) - (\alpha - w_2)(\alpha - w_1) + DQ\alpha(\alpha - w_2)(\alpha - w_1),$
$f_3(w_1, w_2) = k(\alpha - w_1) + l(\alpha - w_2) + DQ(\alpha - w_2)(\alpha - w_1).$

$$I_1 = \alpha_1^{-n}\left\{\frac{1}{D(\alpha-w_2)} - \frac{\alpha_2+\alpha_3}{D(\alpha-w_1)} + \frac{\alpha_2\alpha_3}{D(\alpha-w_0)} + \frac{Q(1-\alpha_2)(1-\alpha_3)}{k+l-2}\right\}$$
$$I_2 = \alpha_2^{-n}\left\{\frac{1}{D(\alpha-w_2)} - \frac{\alpha_1+\alpha_3}{D(\alpha-w_1)} + \frac{\alpha_1\alpha_3}{D(\alpha-w_0)} + \frac{Q(1-\alpha_1)(1-\alpha_3)}{k+l-2}\right\}$$
$$I_3 = \alpha_3^{-n}\left\{\frac{1}{D(\alpha-w_2)} - \frac{\alpha_1+\alpha_2}{D(\alpha-w_1)} + \frac{\alpha_1\alpha_2}{D(\alpha-w_0)} + \frac{Q(1-\alpha_1)(1-\alpha_2)}{k+l-2}\right\}.$$

(iii) $f_1(w_1, w_2) = \tan(k\tan^{-1}(w_2)) + \tan(l\tan^{-1}(w_1)),$
$f_2(w_1, w_2) = 1 - \tan(k\tan^{-1}(w_2))\tan(l\tan^{-1}(w_1)), \quad f_3(w_1, w_2) = -f_1(w_1, w_2)$

$I_1 = \alpha_1^{-n}(\tan^{-1}(w_2) - (\alpha_2 + \alpha_3)\tan^{-1}(w_1) + \alpha_2\alpha_3\tan^{-1}(w_0))$
$I_2 = \alpha_2^{-n}(\tan^{-1}(w_2) - (\alpha_1 + \alpha_3)\tan^{-1}(w_1) + \alpha_1\alpha_3\tan^{-1}(w_0))$
$I_3 = \alpha_3^{-n}(\tan^{-1}(w_2) - (\alpha_1 + \alpha_2)\tan^{-1}(w_1) + \alpha_1\alpha_2\tan^{-1}(w_0)).$
Here

$$\alpha_1 + \alpha_2 + \alpha_3 = k, \qquad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = -l, \qquad \alpha_1\alpha_2\alpha_3 = -1.$$

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